Hybrid System Modelling and Simulation with Dirac Deltas

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Modeling, Simulation, and Design Lab (MSDL)

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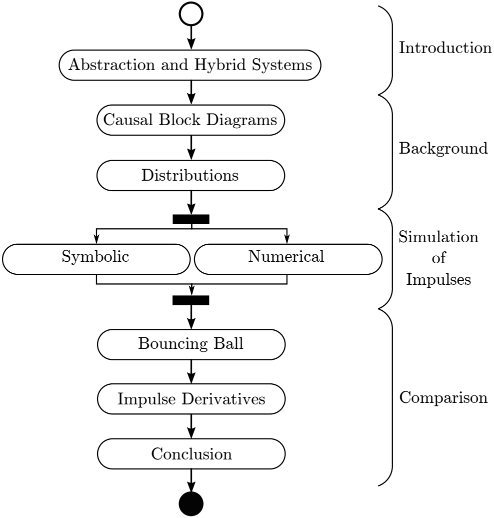
Universiteit Antwerpen

McGill

FLANDERS MAKE

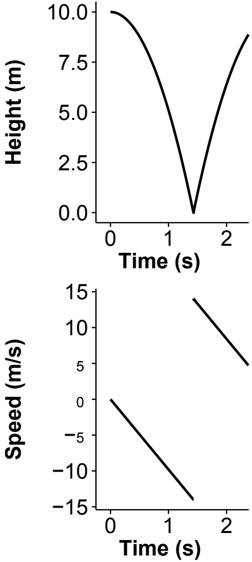
Roadmap


Abstraction and Hybrid Systems
Causal Block Diagrams
Distributions
Symbolic
Numerical
Bouncing Ball
Impulse Derivatives
Conclusion
Introduction
Background
Simulation of Impulses
Comparison

Good Abstractions obey the Laws of Physics

Bouncing ball dynamics:

$$y'' = -g + F_c(t)$$



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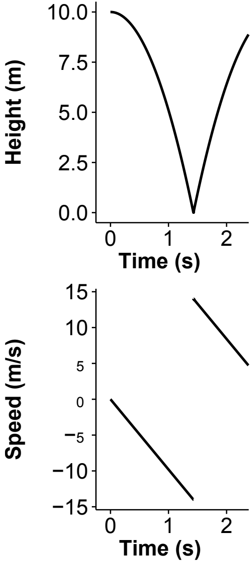
Around a collision $[t_c^-, t_c^+]$:

$$y'(t_c^+) = y'(t_c^-) + \int_{t_c^-}^{t_c^+} -g + F_c(\tau) d\tau$$



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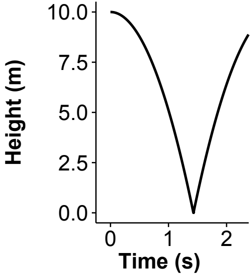
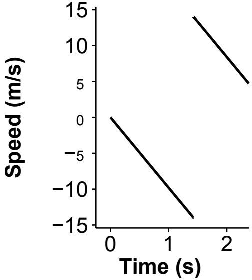
Conservation dictates $y'(t_c^+) = -y'(t_c^-)$
Therefore:

$$\int_{t_c^-}^{t_c^+} F_c(\tau) d\tau = -2y'(t_c^-)$$





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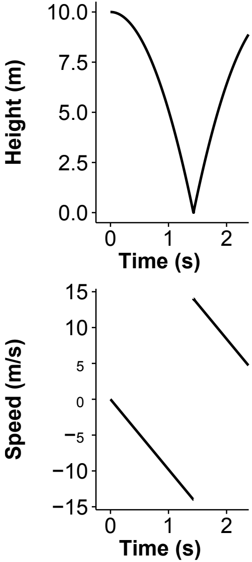
<u>Abstracting the shape of F_c.</u>



Good Abstractions obey the Laws of Physics

Bouncing ball dynamics:

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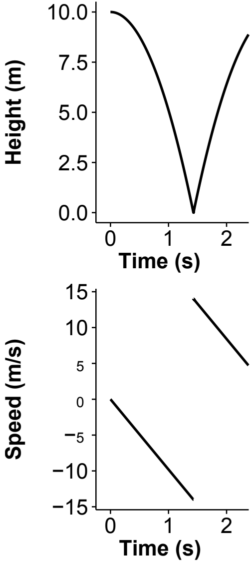
Let δ be a function abstraction, such that:

$$\int_{0^-}^{0^+} \delta(\tau) d\tau = 1$$



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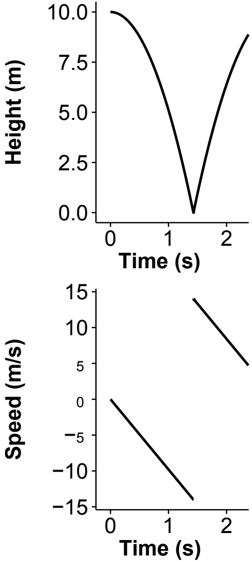
$$F_c(\tau) = -2y'(t_c^-)\delta(t - t_c)$$



Good Abstractions obey the Laws of Physics

Bouncing ball dynamics:

$$y'' = -g - 2y'(t_c^-)\delta(t - t_c)$$

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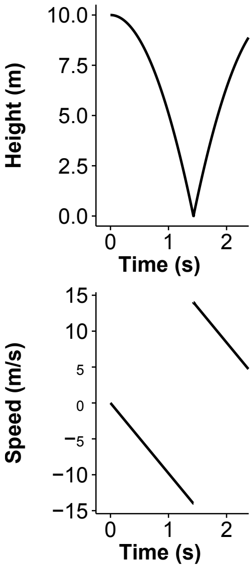
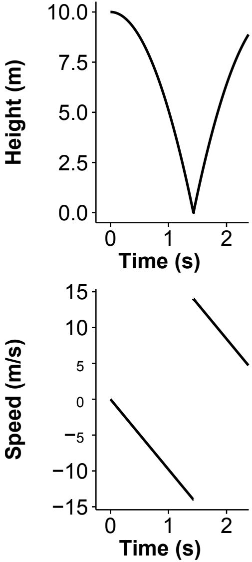


Simulation of Impulses

Separation of Dynamics

- Split the dynamics into piece-wise continuous solutions;
- Solve each one in sequence with traditional numerical methods, properly (respecting Laws of Physics) re-initialize states.

Direct Manipulation

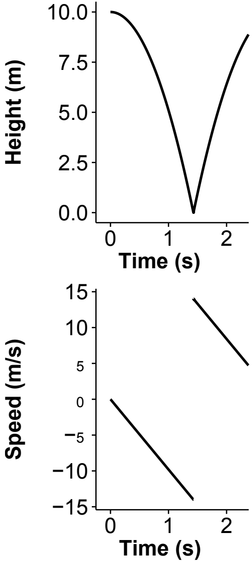


Simulation of Impulses

Separation of Dynamics

$$\begin{pmatrix} y'' & = -g \\ y(0) & = y_0 \\ y'(0) & = v_0 \end{pmatrix} \quad \text{for } 0 \leq t < t_c \text{, and}$$

$$\begin{pmatrix} y'' & = -g \\ y(t_c) & = y(t_c^-) \\ y'(t_c) & = -y'(t_c^-) \end{pmatrix} \quad \text{for } t \geq t_c$$



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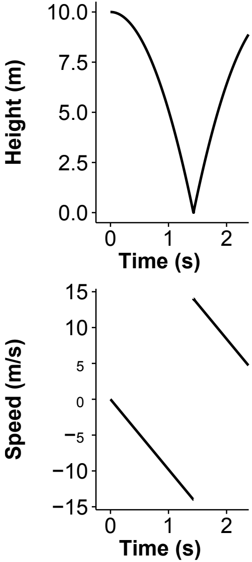
Direct Manipulation

- Compute integration over the impulses.



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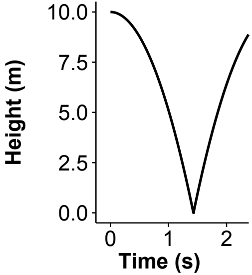
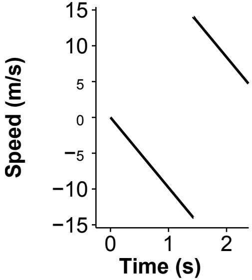




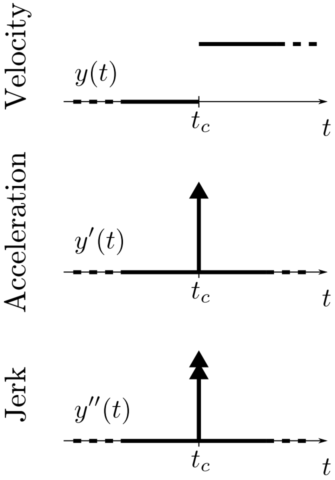
Direct Manipulation of Impulses

Features:

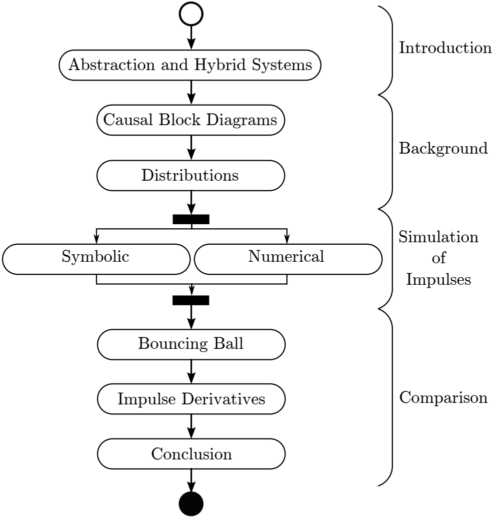
- Handles derivatives of impulses (jerk, snap, crackle, pop,...)

Example:

$$y^{(n)} = \delta^{(n-1)}(t - t_c)$$
$$\text{for } n > 1$$



Recap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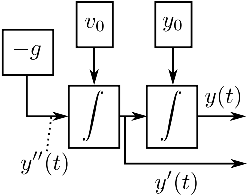
Causal Block Diagrams

$$y'' = -g$$



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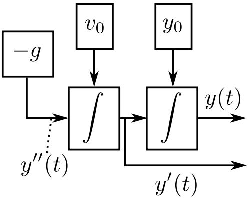

procedure CBDSimulator(Flat D, end_condition)
    step ← 0
    while not end_condition **do**
        schedule ← LoopDetect(DepGraph(D))
        for gblock **in** schedule **do**
            Compute(gblock)
        end for
        step ← step +1
    end while
end procedure

Diving into Dirac Deltas – Distributions

A distribution is a function identified by the way it interacts with other *test* functions, and not by its shape.

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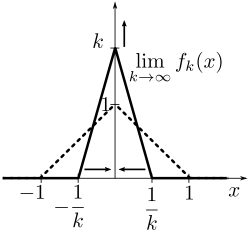
$$\delta(x) = \lim_{k\to\infty} \begin{cases} \max(0, k + k^2 x) \text{ if } x \leq 0 \\ \max(0, k - k^2 x) \text{ otherwise} \end{cases}$$



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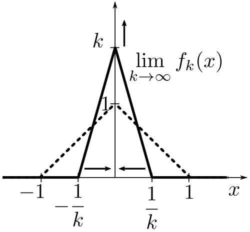
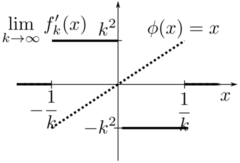
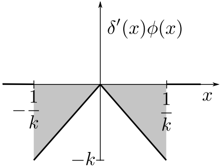


Properties:

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0)$$

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)dx = -\phi'(0)$$





Diving into Dirac Deltas – Distributions

Two distributions are equal if *the result of their interactions with any smooth function is equal*:

$$f = g \iff \underbrace{\int f(x)\phi(x)dx}_{\langle f,\phi\rangle} = \underbrace{\int g(x)\phi(x)dx}_{\langle g,\phi\rangle} \text{ for all } \phi$$

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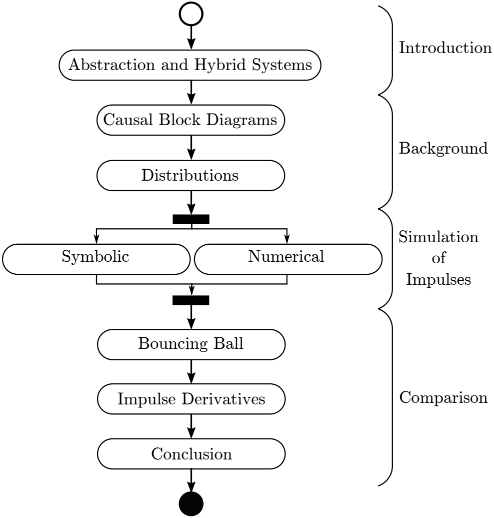
Example:

$$\left\langle H'(x), \phi \right\rangle = \int_{-\infty}^{\infty} H'(x)\phi(x)dx$$

$$= \left[H(x)\phi(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)\phi'(x)dx$$

$$= -\int_{0}^{\infty} \phi'(x)dx = -\left[\phi(x)\right]_{0}^{\infty} = \phi(0) = \langle \delta(x), \phi \rangle$$

Recap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Causal Block Diagrams
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Symbolic
Numerical
Bouncing Ball
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Introduction
Background
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Symbolic Manipulation

Signal Representation:

$$S(t) = s(t) + \sum_{i=0}^{n} \sum_{\tau_j \in \{\tau_j\}} a_{ij}\delta^{(i)}(t - \tau_j)$$

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$$S(t_i) \in \mathbb{R}^2 \times \mathbb{R}^m$$

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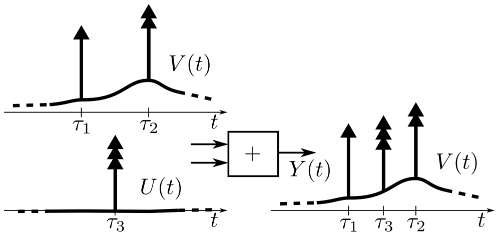
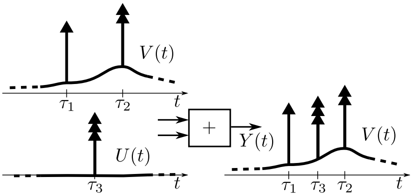


Symbolic Manipulation – Sum



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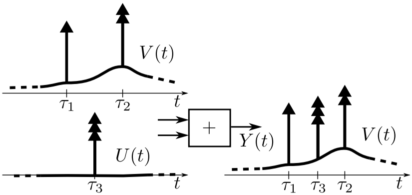


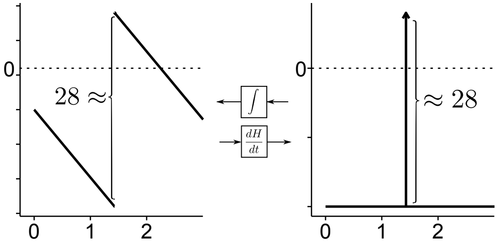
$$Y(t) = U(t) + V(t) \Leftrightarrow$$

$$\langle Y(t), \varphi(t) \rangle = \langle U(t) + V(t), \varphi(t) \rangle \qquad \text{for any test function } \varphi$$

$$= \langle U(t), \varphi(t) \rangle + \langle V(t), \varphi(t) \rangle$$

$$= \left\langle u(t) + v(t) + \sum_{i=0}^{n_u} \sum_{\tau_j^u \in \left\{ \tau_j^u \right\}} a_{ij} \delta^{(i)}(t - \tau_j^u) + \sum_{i=0}^{n_v} \sum_{\tau_j^v \in \left\{ \tau_j^v \right\}} b_{ij} \delta^{(i)}(t - \tau_j^v), \varphi(t) \right\rangle$$

Symbolic Manipulation – Integral and Derivative



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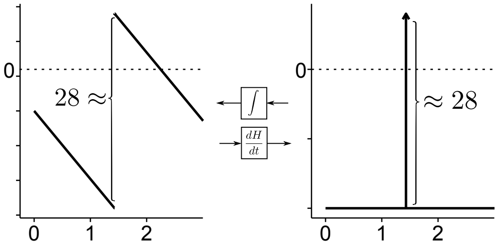


$$\left\langle \int_0^t U(x)dx, \varphi(t) \right\rangle =$$

$$\left\langle \int_0^t u(x)dx + \sum_{\tau_j \in \{\tau_j\}} a_{0j} H(x - \tau_j) + \sum_{i=1}^{n_u} \sum_{\tau_j \in \{\tau_j\}} a_{ij} \delta^{(i-1)}(t - \tau_j), \varphi(t) \right\rangle$$

$$\langle U'(t), \varphi(t) \rangle =$$

$$\left\langle u'(t) + \sum_{t_d \in \{t_d\}} (u(t_d^+) - u(t_d^-))\delta(t - t_d) + \sum_{i=0}^{n_u} \sum_{\tau_j \in \{\tau_j\}} a_{ij} \delta^{(i+1)}(t - \tau_j)(t), \varphi(t) \right\rangle$$

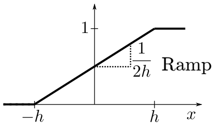
Numerical Approximation of Impulses

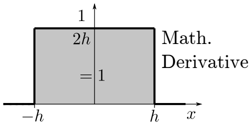
Start from:

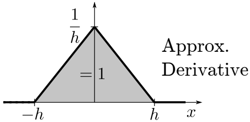
$$\delta(x) = \lim_{k \to \infty} H_k'(x) = \lim_{k \to \infty} \begin{cases} \frac{1}{2}k \text{ if } -\frac{1}{k} \leq x \leq \frac{1}{k} \\ 0 \text{ otherwise} \end{cases}$$

with

$$H_k(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{k} \\ \frac{1}{2} + \frac{1}{2}kx & \text{if } -\frac{1}{k} \leq x \leq \frac{1}{k} \\ 1 & \text{if } x > \frac{1}{k} \end{cases}$$







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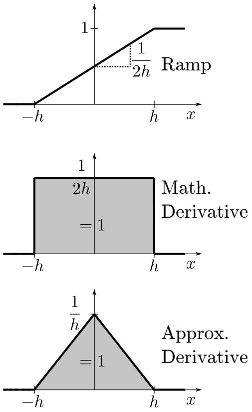
With derivative approximation:

$$H_{1/h}'(\tau_d) \approx \frac{H_{1/h}(\tau_d) - H_{1/h}(\tau_d - h)}{h} \approx \frac{1}{h}$$



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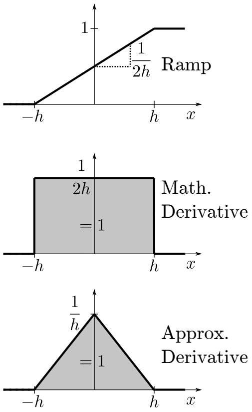
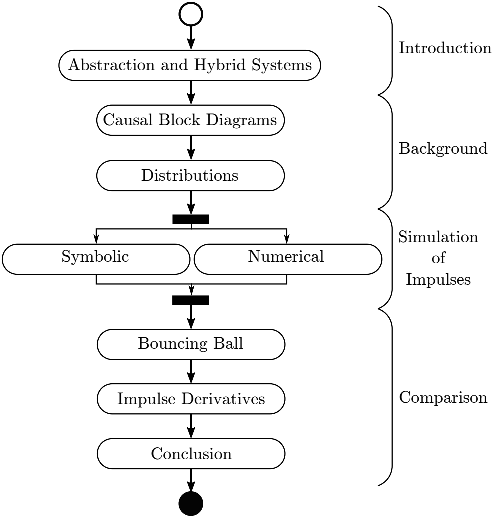
Choose:

$$\delta(t - \tau_d) \approx H_{1/h}'(t - \tau_d)$$



Recap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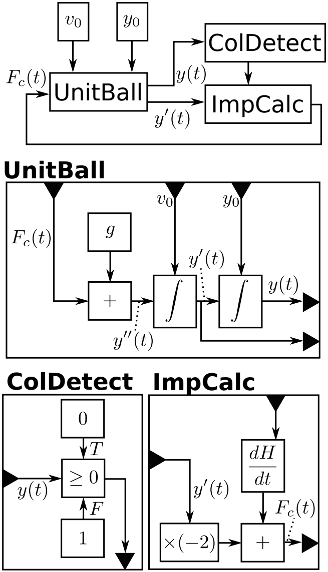
Comparison – Without Impulse Derivatives

Example: Bouncing ball



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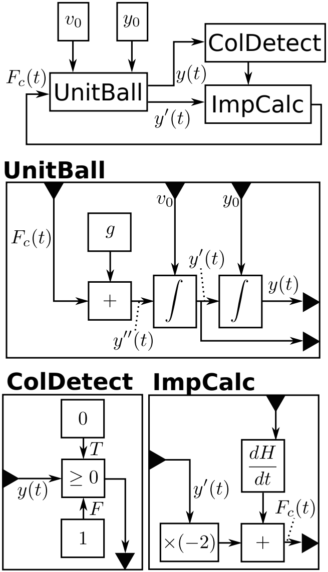
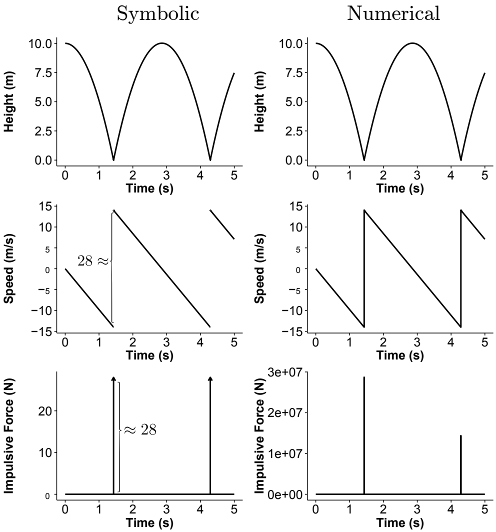
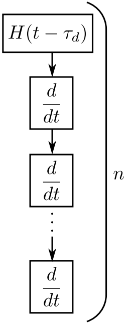
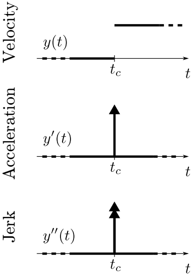




Comparison – With Impulse Derivatives

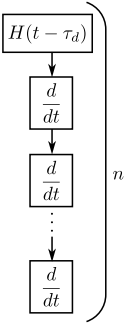
$$S(t) = H(t - \tau_d)$$
$$S'(t) = \delta(t - \tau_d)$$
$$S^{(2)}(t) = \delta'(t - \tau_d)$$
$$\dots$$
$$S^{(n+1)}(t) = \delta^{(n)}(t - \tau_d)$$

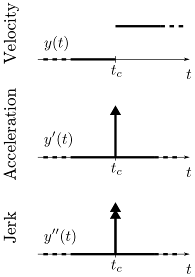




Comparison – With Impulse Derivatives

$$S(t) = H(t - \tau_d)$$
$$S'(t) = \delta(t - \tau_d)$$
$$S^{(2)}(t) = \delta'(t - \tau_d)$$
$$\dots$$
$$S^{(n+1)}(t) = \delta^{(n)}(t - \tau_d)$$





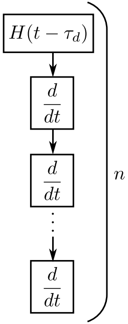
Time	$S(t)$	$S'(t)$	$S^{(2)}(t)$	$S^{(3)}(t)$	...	$S^{(n-1)}(t)$	$S^{(n)}(t)$
$\tau_d - h$	0	0	0	0	...	0	0
τ_d	1	$1/h$	$1/h^2$	$1/h^3$	...	$1/h^n$	$1/h^n$
$\tau_d + h$	1	0	$-1/h^2$	$-2/h^3$	...	$-(n-2)/h^n$	$-(n-1)/h^n$
$\tau_d + 2h$	1	0	0	$1/h^3$	...	$\binom{n-2}{2}/h^n$	$\binom{n-1}{2}/h^n$
$\tau_d + 3h$	1	0	0	0	...	$-\binom{n-2}{3}/h^n$	$-\binom{n-1}{3}/h^n$
...	...	...	...	...	...	...	...
$\tau_d + (n-1)h$	1	0	0	0	...	0	$(-1)^{n-1}\binom{n-1}{n-1}/h^n$
$\tau_d + nh$	1	0	0	0	...	0	0

Comparison – Impulse Derivatives

Differences:

- Numerical solution shifted by $n \times h$ time units;
- Maximum magnitude for a discontinuity D:

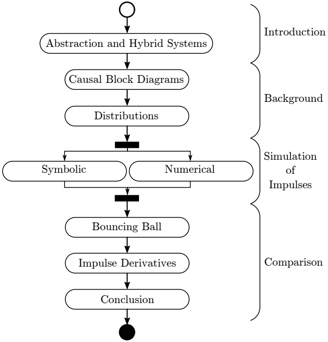
$$D\binom{n-1}{k-1}/h^k \quad \text{where} \quad k = floor\left(\frac{n}{2}\right)$$



Time	$S(t)$	$S'(t)$	$S^{(2)}(t)$	$S^{(3)}(t)$	...	$S^{(n-1)}(t)$	$S^{(n)}(t)$
$\tau_d - h$	0	0	0	0	...	0	0
τ_d	1	$1/h$	$1/h^2$	$1/h^3$	...	$1/h^n$	$1/h^n$
$\tau_d + h$	1	0	$-1/h^2$	$-2/h^3$	...	$-(n-2)/h^n$	$-(n-1)/h^n$
$\tau_d + 2h$	1	0	0	$1/h^3$	...	$\binom{n-2}{2}/h^n$	$\binom{n-1}{2}/h^n$
$\tau_d + 3h$	1	0	0	0	...	$-\binom{n-2}{3}/h^n$	$-\binom{n-1}{3}/h^n$
...	...	...	...	...	...	...	...
$\tau_d + (n-1)h$	1	0	0	0	...	0	$(-1)^{n-1}\binom{n-1}{n-1}/h^n$
$\tau_d + nh$	1	0	0	0	...	0	0

Conclusion

- For models that contain no impulse derivatives, both approaches are equivalent
- Otherwise, numerical approach is less accurate:
  - Delays signals
  - Computes large values



Thank you!