# Demo: Stabilization Technique in INTO-CPS

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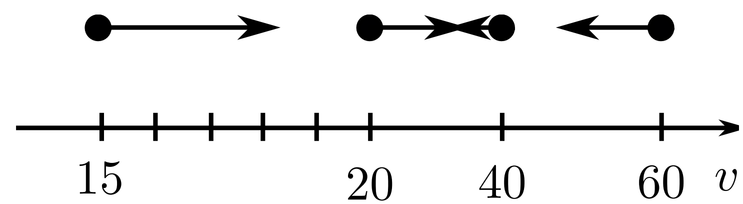
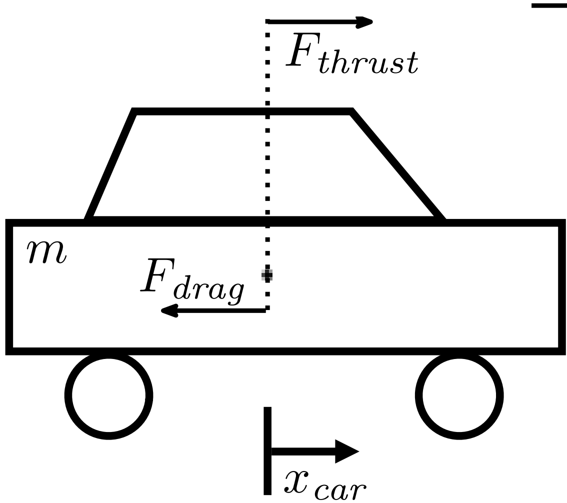
# Example IVP

$$\dot{x} = f(x, u) \quad \text{with} \quad x(0) = x_0$$

Example:

$$\dot{v} = \frac{1}{m}\left[\underbrace{k(v_d - v)}_{F_{thrust}} \underbrace{- dv}_{F_{drag}}\right] \quad \text{with} \quad v(0) = v_0$$



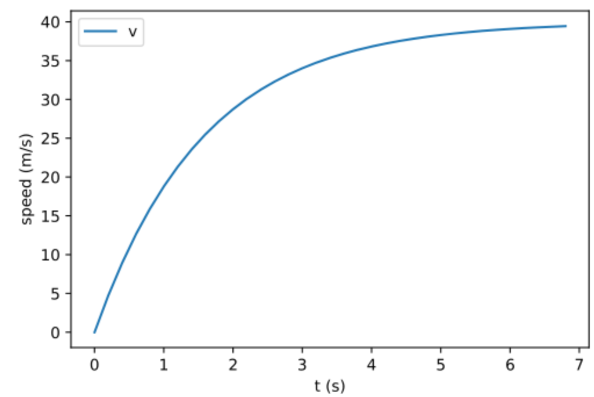




$k = 10^3 \quad m = 1576(kg)$

$d = 0.5 \quad v_d = 40(m/s)$

# Stability of ODEs

Do all solutions of given IVP tend to an equilibrium?
Formally, for any solution $\boldsymbol{x}(t)$, does $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$?




Application: Does the cruise control drive the car to a stable velocity?

$$\dot{v} = \frac{1}{m}\left[k(v_d - v) - dv\right] \quad \text{with} \quad v(0) = v_0$$

Stable velocity: $v_t = (kv_d)/(k+d)$

Let $a = -(1/m)(k+d)$ and $b = (1/m)(kv_d)$ so that $\dot{v} = av + b$

Then, introduce new variable $\bar{v} = v - v_t$ so that $\dot{\bar{v}} = a\bar{v}$

Solution is $\bar{v}(t) = e^{at}\bar{v}_0$

Since $a < 0$, $\bar{v}(t) \to 0$ as $t \to \infty$

# Stability of Ordinary Differential Equations

For any solution $\boldsymbol{x}(t)$, does $\lim_{t \to \infty} \|\boldsymbol{x}(t)\| = 0$?

Scalar linear ODEs: $\dot{x} = ax$ with $x(0) = x_0$

Stable if $a < 0$

Vector linear ODEs: $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ with $\boldsymbol{x}(0) = \boldsymbol{x_0}$

Stable if $\forall \lambda \in \mathrm{Eig}(A),\ \mathbb{R}\mathrm{e}\{\lambda\} < 0$

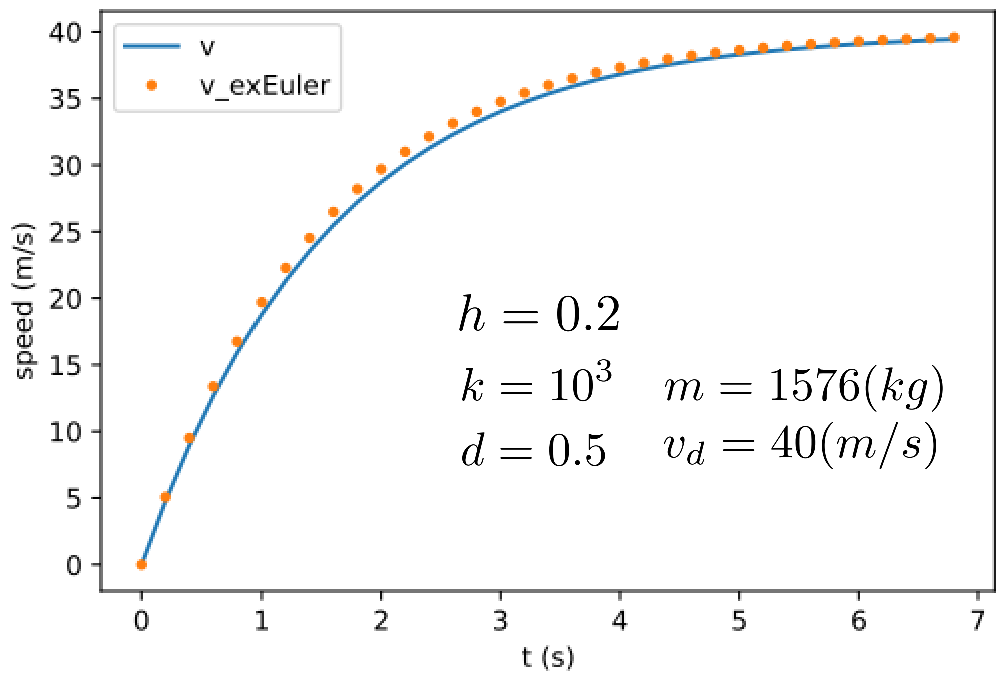
# Scalar Initial Value Problems - Approximation

$$\dot{x} = f(x, u) \quad \text{with} \quad x(0) = x_0$$

*Explicit Euler Method*:

$$x(t + h) \approx x(t) + f(x(t), u(t))h \quad \text{with} \quad x(0) = x_0$$

| v | dv_dt |
|---|---|
| 0.0 | 25.380 |
| 5.076 | 22.158 |
| 9.508 | 19.345 |



# Numerical Stability of Euler Method

Scalar linear ODEs: $\dot{x} = ax$

Explicit Euler Method: $\boldsymbol{x}(t+h) \approx \boldsymbol{x}(t) + F(\boldsymbol{x}(t), \boldsymbol{u}(t))h$

$$x(t+h) \approx x(t) + ahx(t) = (1+ah)x(t) = (1+ah)^n x(0)$$

$$\lim_{n\to\infty} (1+ah)^n x(0) = 0 \text{ if } |1+ah| < 1$$

Vector linear ODEs: $\dot{\boldsymbol{x}} = A\boldsymbol{x}$

$$\lim_{n\to\infty} (I+Ah)^n \boldsymbol{x}(0) = 0 \text{ if } \rho(I+Ah) < 1$$

$$\rho(M) = \max_{\lambda \in \mathrm{Eig}(M)} |\lambda|$$

# Stability Analysis Summary

1. Check if original system is stable.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \quad \forall \lambda \in \mathrm{Eig}(A),\ \mathbb{R}\mathrm{e}\{\lambda\} < 0$$

2. Apply numerical method equation to original system

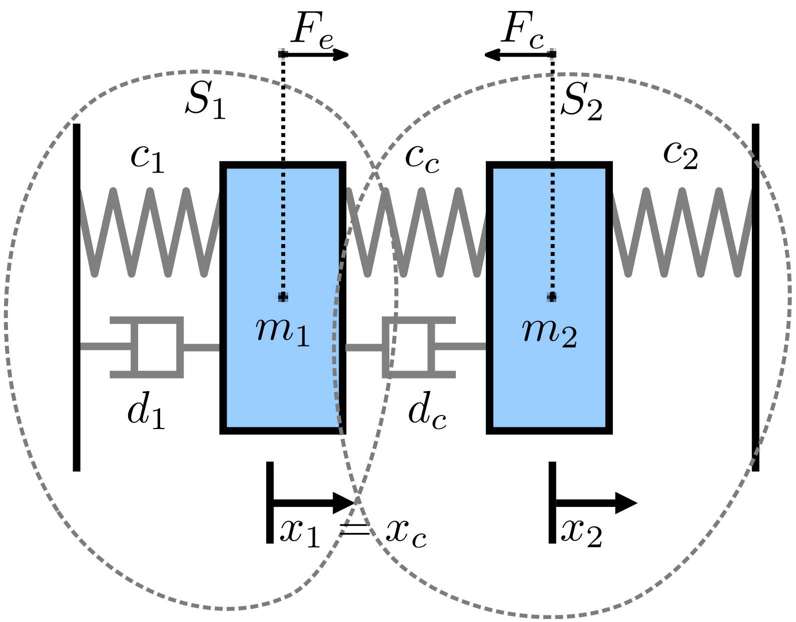
$$\boldsymbol{x}(t+h) \approx \tilde{A}\boldsymbol{x}(t)$$

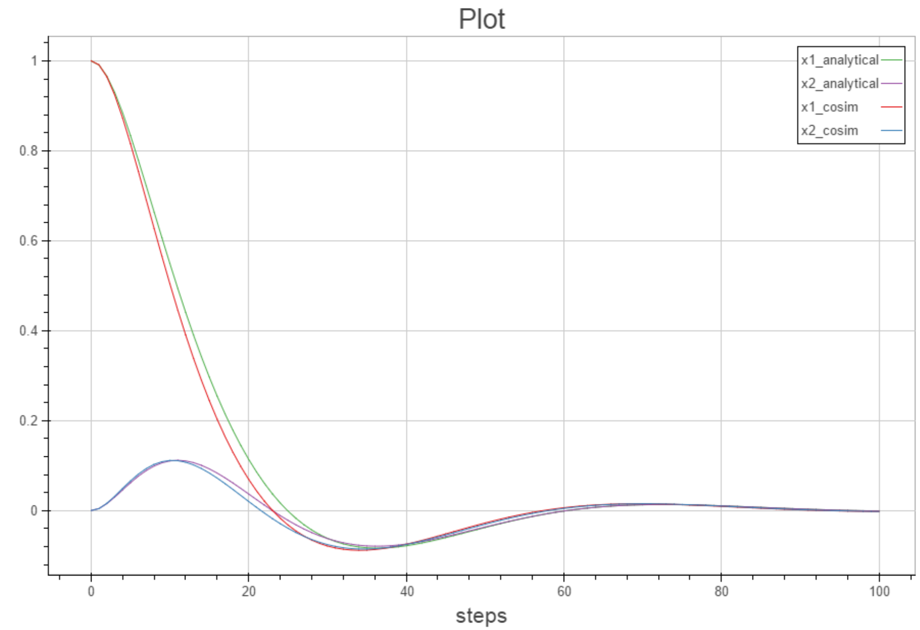
equation

$$\rho(\tilde{A}) < 1 \qquad \rho(M) = \max_{\lambda \in \mathrm{Eig}(M)} |\lambda|$$

3. Check if it is stable.

# Application to Co-simulation




$$\dot{x_1} = v_1; \qquad m_1 \cdot \dot{v_1} = -c_1 \cdot x_1 - d_1 \cdot v_1 + F_e$$

$$x_1(0) = p_1; \qquad v_1(0) = s_1$$

$$\dot{x_2} = v_2$$

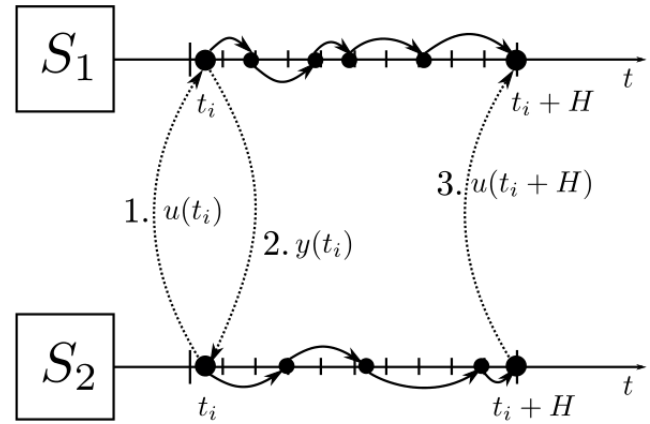
$$m_2 \cdot \dot{v_2} = -c_2 \cdot x_2 - F_c$$

$$F_c = c_c \cdot (x_2 - x_c) + d_c \cdot (v_2 - \dot{x_c})$$

$$x_2(0) = p_2$$

$$v_2(0) = s_2$$

# Application to Co-sim.




Cosim unit:

$$\dot{x}_j = A_j x_j + B_j u_j$$
$$y_j = C_j x_j + D_j u_j$$

Target form:

$$\begin{bmatrix} \tilde{x}_j(t_i + H) \\ \tilde{u}_j(t_i + H) \end{bmatrix} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t_i) \\ u_j(t_i) \end{bmatrix}$$

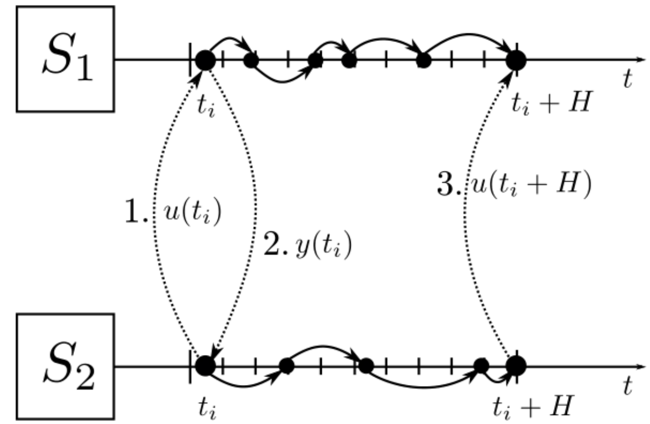
Zero order hold:

$$\tilde{u}_j(t) = u_j(t_i), \text{ for } t \in [t_i, t_{i+1})$$

Cosim unit:

$$\begin{bmatrix} \dot{x}_j \\ \dot{\tilde{u}}_j \end{bmatrix} = \begin{bmatrix} A_j & B_j \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_j \\ \tilde{u}_j \end{bmatrix} \qquad \tilde{u}_j(t_i) = u_j(t_i)$$

# Application to Co-sim.




Cosim unit:

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Target form:

$$\begin{bmatrix} \tilde{x}_j(t_i + H) \\ \tilde{u}_j(t_i + H) \end{bmatrix} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t_i) \\ u_j(t_i) \end{bmatrix}$$

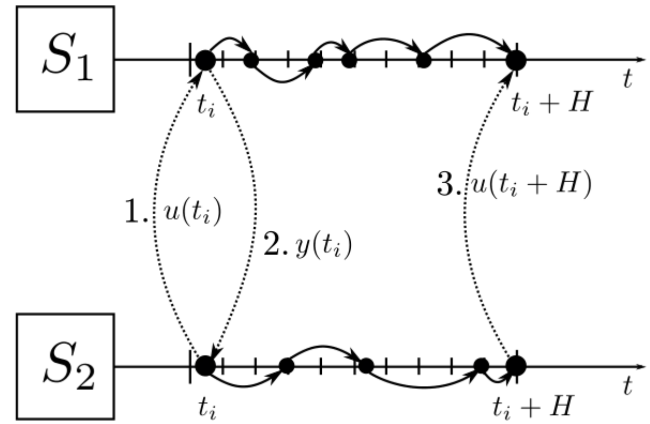
Cosim unit with internal iteration:

$$\begin{bmatrix} \tilde{x}_j(t_{i+1}) \\ \tilde{u}_j(t_{i+1}) \end{bmatrix} = \tilde{A}_j^{k_j} \begin{bmatrix} \tilde{x}_j(t_i) \\ \tilde{u}_j \end{bmatrix}$$

e.g., Fw. Euler:

$$\tilde{A}_j = \mathbf{I} + \begin{bmatrix} A_j & B_j \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

# Application to Co-sim.




Cosim unit:

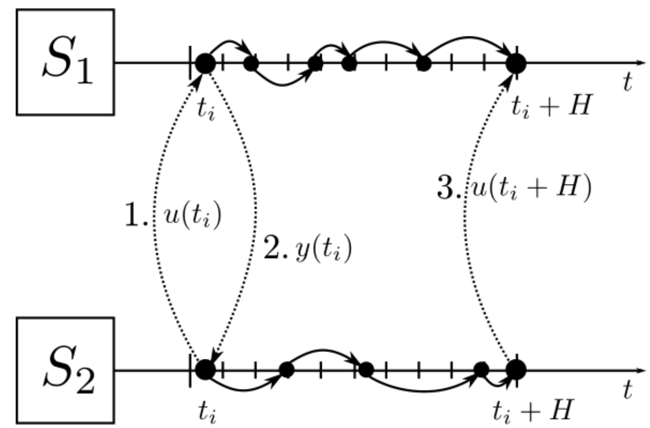
$$\begin{bmatrix} \tilde{x}_j(t_{i+1}) \\ \tilde{u}_j(t_{i+1}) \end{bmatrix} = \tilde{A}_j^{k_j} \begin{bmatrix} \tilde{x}_j(t_i) \\ \tilde{u}_j \end{bmatrix} \quad \tilde{u}_j(t_i) = u_j(t_i)$$

Target form:

$$\begin{bmatrix} \tilde{x}_j(t_i + H) \\ \tilde{u}_j(t_i + H) \end{bmatrix} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t_i) \\ u_j(t_i) \end{bmatrix}$$

$$\tilde{A}_j^{k_j} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix}$$

# Application to Co-sim.




Cosim unit:

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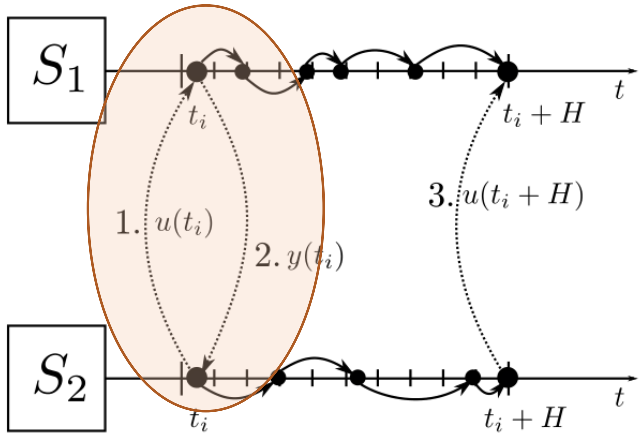
$$\tilde{u}_j(t_i) = u_j(t_i)$$

Ideal coupling cosim:

$$u_1(t) = y_2(t) = C_2 \tilde{x}_2(t)$$
$$u_2(t) = y_1(t) = C_1 \tilde{x}_1(t) + D_1 u_1(t)$$

# Jacobi Coupling

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Actual cosim coupling:

$$u_1(t_i) = C_2 \tilde{x}_2(t_i)$$
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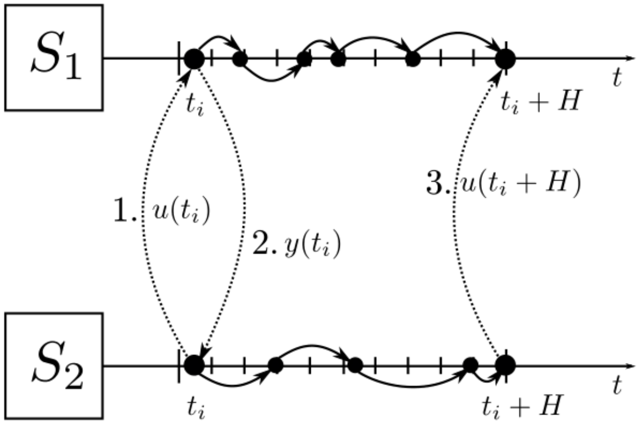


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Actual cosim coupling:

$$\tilde{u}_j(t_i) = u_j(t_i)$$

$$u_1(t_i) = C_2 \tilde{x}_2(t_i)$$
$$u_2(t_i) = C_1 \tilde{x}_1(t_i) + D_1 \tilde{u}_1(t_i)$$

$$\tilde{x}_1(t_{i+1}) = M_{1,x_1} \tilde{x}_1(t_i) + M_{1,u_1} C_2 \tilde{x}_2(t_i)$$
$$\tilde{u}_1(t_{i+1}) = M_{2,x_1} \tilde{x}_1(t_i) + M_{2,u_1} C_2 \tilde{x}_2(t_i)$$
$$\tilde{x}_2(t_{i+1}) = M_{1,u_2} C_1 \tilde{x}_1(t_i) + M_{1,u_2} D_1 \tilde{u}_1(t) + M_{1,x_2} \tilde{x}_2(t_i)$$
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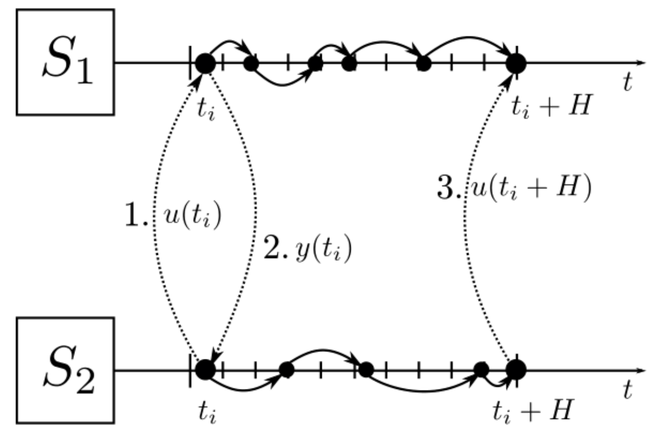


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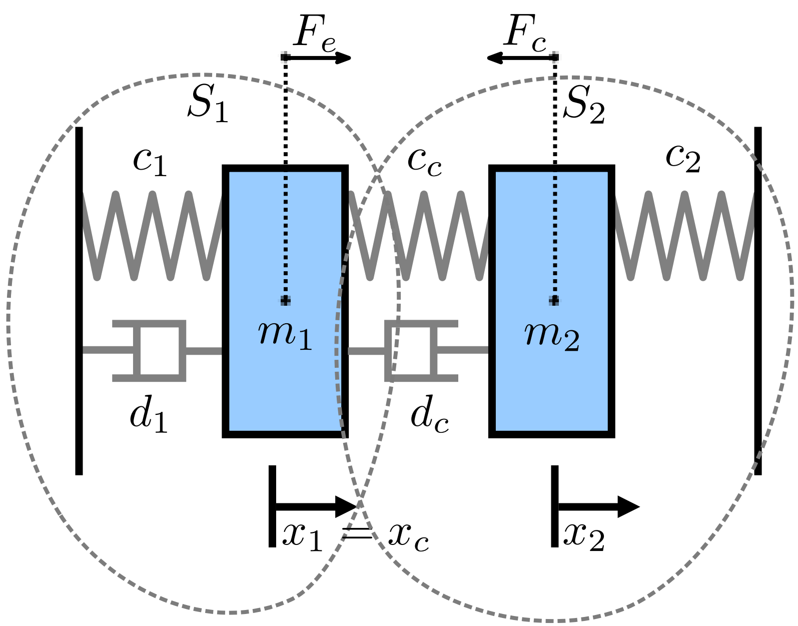
Actual cosim coupling: $\tilde{u}_j(t_i) = u_j(t_i)$

$$u_1(t_i) = C_2 \tilde{x}_2(t_i)$$

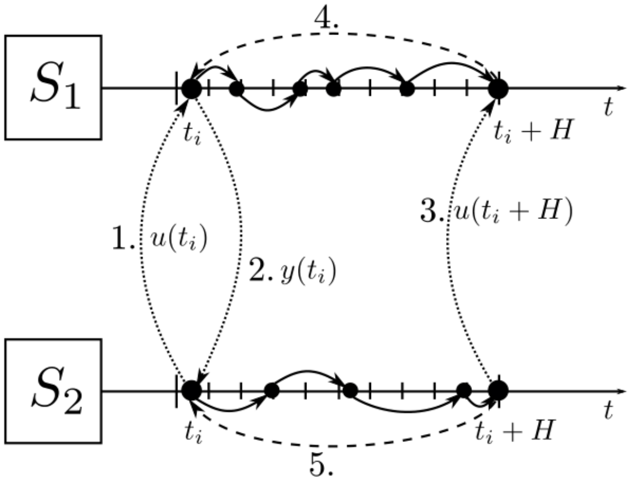
$$u_2(t_i) = C_1 \tilde{x}_1(t_i) + D_1 \tilde{u}_1(t_i)$$

$$\begin{bmatrix} \tilde{x}_1(t_{i+1}) \\ \tilde{u}_1(t_{i+1}) \\ \tilde{x}_2(t_{i+1}) \\ \tilde{u}_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} M_{1,x_1} & 0 & M_{1,u_1} C_2 & 0 \\ M_{2,x_1} & 0 & M_{2,u_1} C_2 & 0 \\ M_{1,u_2} C_1 & M_{1,u_2} D_1 & M_{1,x_2} & 0 \\ M_{2,u_2} C_1 & M_{2,u_2} D_1 & M_{2,x_2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t_i) \\ \tilde{u}_1(t_i) \\ \tilde{x}_2(t_i) \\ \tilde{u}_2(t_i) \end{bmatrix}$$

# Jacobi Coupling - MSD



# Iterative Coupling

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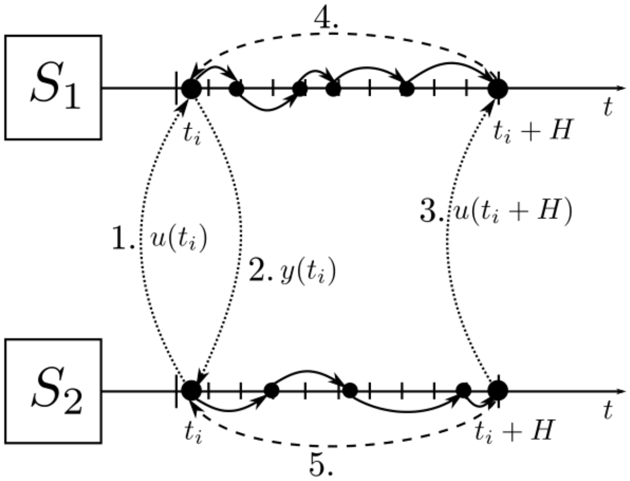


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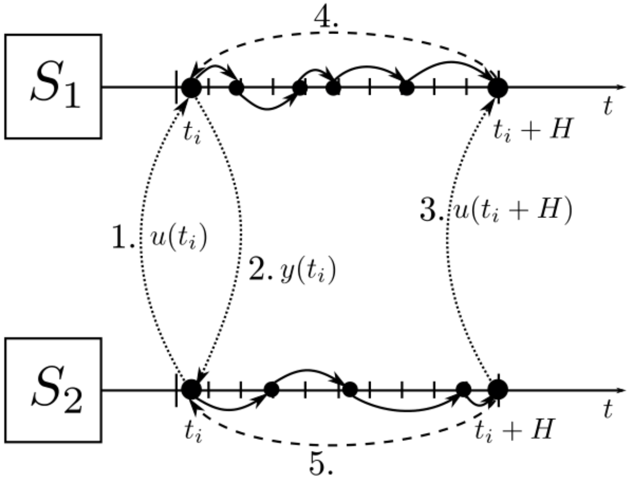


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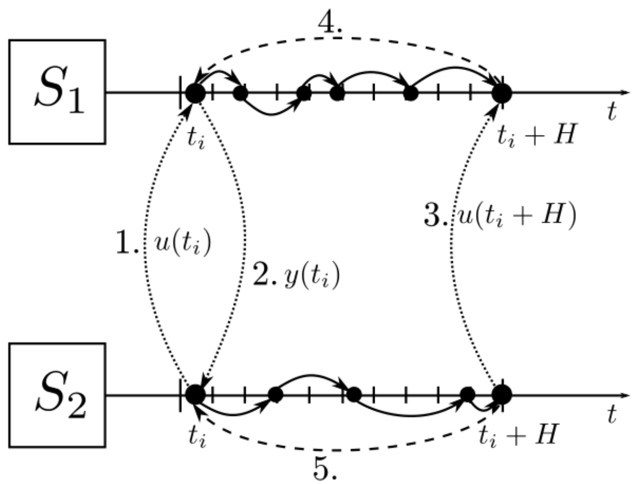


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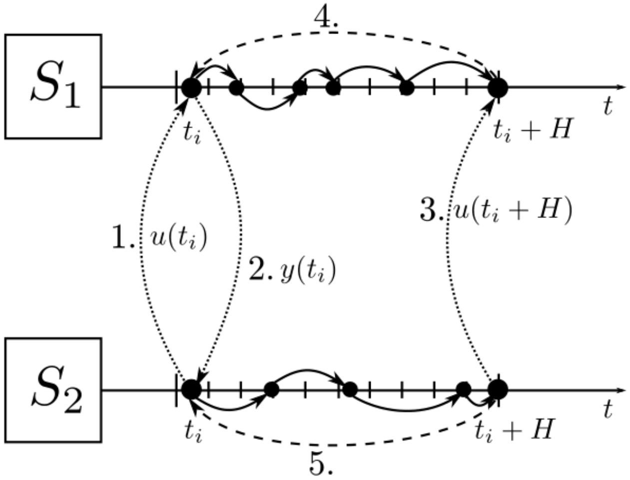
# Iterative Coupling

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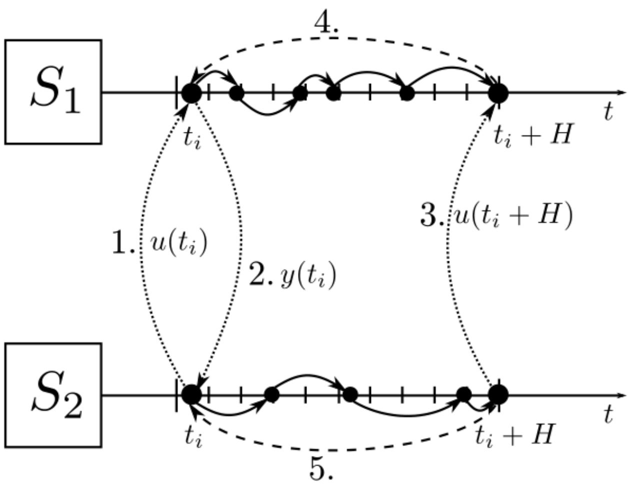


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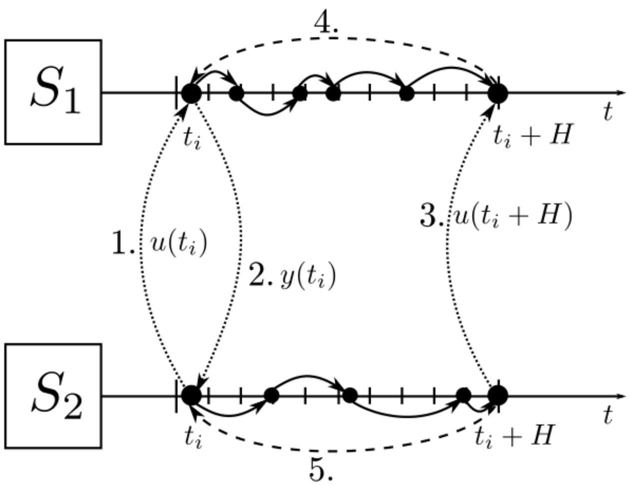
$$\begin{bmatrix} \tilde{x}_j(t_i + H) \\ \tilde{u}_j(t_i + H) \end{bmatrix} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t_i) \\ u_j(t_i) \end{bmatrix}$$

$$\tilde{u}_j(t_i) = u_j(t_i)$$

$$\begin{bmatrix} \tilde{x}_1(t_{i+1}) \\ u_1(t_{i+1}) \\ \tilde{x}_2(t_{i+1}) \\ u_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} M_{1,x_1} & 0 & 0 & 0 \\ M_{2,x_1} & 0 & 0 & 0 \\ 0 & 0 & M_{1,x_2} & 0 \\ 0 & 0 & M_{2,x_2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t_i) \\ u_1(t_i) \\ \tilde{x}_2(t_i) \\ u_2(t_i) \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & M_{1,u_1} C_2 & 0 \\ 0 & 0 & M_{2,u_1} C_2 & 0 \\ M_{1,u_2} C_1 & M_{1,u_2} D_1 & 0 & 0 \\ M_{2,u_2} C_1 & M_{2,u_2} D_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t_{i+1}) \\ u_1(t_{i+1}) \\ \tilde{x}_2(t_{i+1}) \\ u_2(t_{i+1}) \end{bmatrix}$$

# Iterative Coupling




Cosim unit:

$$\begin{bmatrix} \tilde{x}_j(t_i + H) \\ \tilde{u}_j(t_i + H) \end{bmatrix} = \begin{bmatrix} M_{1,x_j} & M_{1,u_j} \\ M_{2,x_j} & M_{2,u_j} \end{bmatrix} \begin{bmatrix} \tilde{x}_j(t_i) \\ u_j(t_i) \end{bmatrix}$$

$$\tilde{u}_j(t_i) = u_j(t_i)$$

$$\bar{x}_{i+1} = \bar{M}_i \bar{x}_i + \bar{M}_{i+1} \bar{x}_{i+1}$$

$$\bar{x}_{i+1} = (I - \bar{M}_{i+1})^{-1} \bar{M}_i \bar{x}_i$$

# Iterative Coupling - MSD