Petri Net Analysis

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Overview

- Behavioral Properties
- Incidence Matrix & State Equations
- Analysis of Marked Graphs

Behavioral Properties

- Properties dependent of initial marking

1. Reachability
   - Marking M reachable from M_o if ∃ a sequence of firing from M_o to M
   - Define $R(M_o)$ to be set of marking reachable from M_o

Behavioral Properties (cont)

2. Boundedness

   - Net bounded if num tokens in each place not exceed finite num k for any marking reachable from M_o

Behavioral Properties (cont)

3. Liveness
   - L0: t can never be fired in any firing sequence
   - L1: t can be fired at least once in some firing sequence
   - L2: t can be fired at least k times in some firing sequence
   - L3: t appears ∞ in some firing seq.
   - L4: t is L1-live for every marking

Behavioral Properties (cont)

4. Reversibility
   - For each marking M in $R(M_o)$, M_o reachable from M

5. Coverability
   - $\exists$ marking M′ in $R(M_o)$ such that $M'(p) \geq M(p)$ for each place p

Behavioral Properties (cont)

6. Persistence
   - For any 2 enabled transitions, the firing of one will not disable the other

Behavioral Properties (cont)

7. Synchronic distance
   - metric closely related to degree of mutual dependence between 2 events in a condition/event syst.

   $d_{12} = \max |\text{num } t_1 - \text{num } t_2|$
   num t_1: num times t_1 fires in firing seq starting at any marking

Behavioral Properties (cont)

8. Fairness

   - Bounded-fairness

     2 transition are in bounded-fair relation if max num time that either can fire while other not firing is bounded

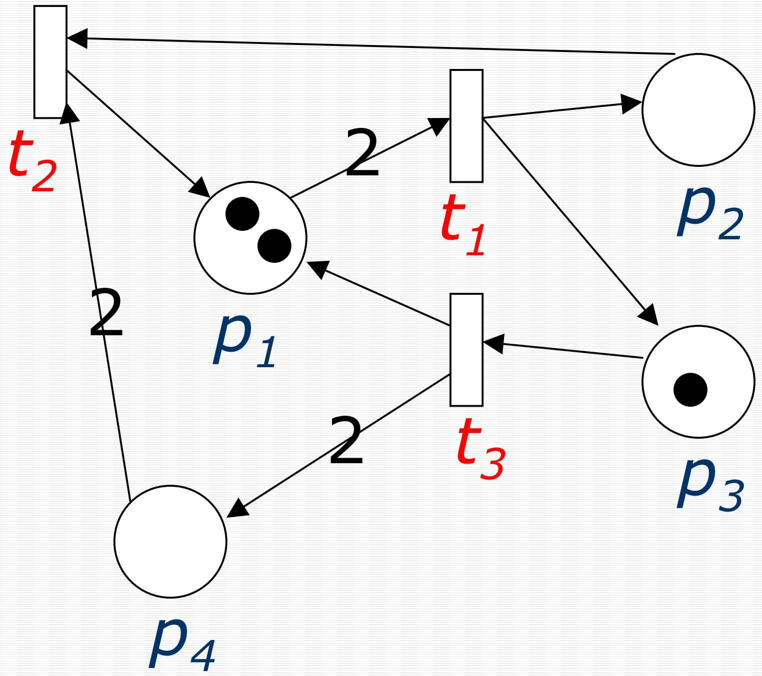
Behavioral Properties (cont)

- Unconditionally fair

  Firing seq is uncond. fair if finite or every transition in net appears infinitely often

Incidence Matrix

- n x m matrix A (n trans. & m places) where $a_{ij} = a_{ij}^{+} - a_{ij}^{-}$



$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

State Equation

- $M_k = M_{k-1} + A^T u_k \qquad k = 1,2,\ldots$

Example:

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

State Equation (cont)

- $M_d = M_0 + A^T \Sigma^d{}_{k=1} u_k$
  $\Rightarrow A^T x = \Delta M$
- Let r be the rank of A, partition A

$$A = \begin{array}{c} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \\ \begin{matrix} m-r & \quad r \end{matrix} \end{array}$$

Circuit matrix: $B_f = [\ I_\mu : -\ A^T{}_{11}(A^T{}_{12})^{-1}]$

State Equation (cont)

- Example

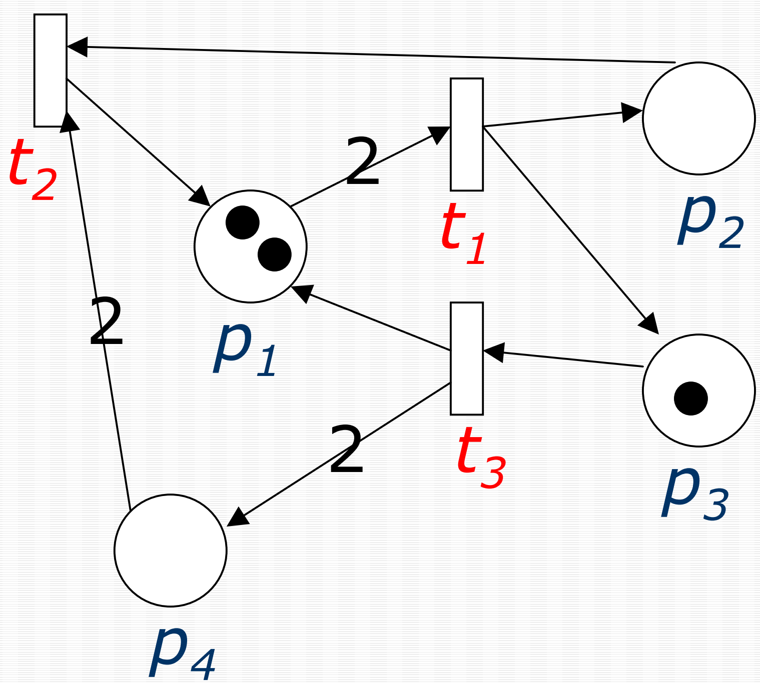
rank = 2

$$A = \left[\begin{array}{cc:cc} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & -2 \\ \hdashline 1 & 0 & -1 & 2 \end{array}\right] \qquad B_f = \begin{bmatrix} 1 & 0 & 2 & ½ \\ 0 & 1 & -1 & -½ \end{bmatrix}$$

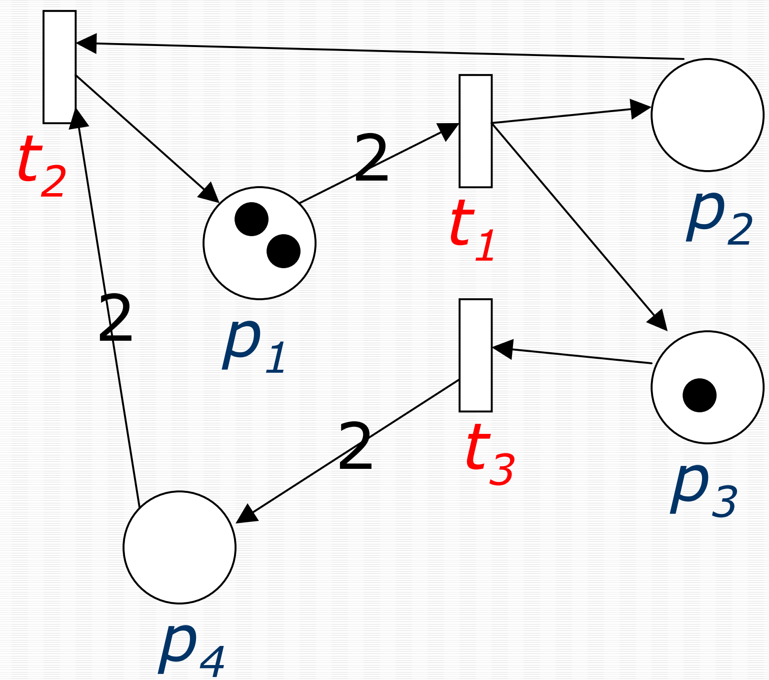
Marked Graph

- Analyzing matrix equations applicable only to special subclasses of Petri nets
- Look at marked graphs
  - Petri net such that each place p has exactly one input transition & exactly one output transition

Marked Graph (cont)



Not marked graph



Marked graph

Analysis of Marked Graph

- Reachability
- Weighted sum of tokens
- Token distance

Reachability

- M_d is reachable from M_0 iff
  $B_f M_0 = B_f M_d$ (B_f: circuit matrix)

  & ∃ such firing sequence
  Ie. $A^T x = 0$ is solvable for x

Weighted Sum of Tokens

- Often interested in finding max or min weighted sum of tokens (M^TW)

  $\max \{M^TW \mid M \in R(M_0)\}$
  $\quad = \min \{M^T{}_0 I \mid I \geq W, AI = 0\}$
  vice-versa

  where W is mx1 matrix whose i^{th} entry is num token in place i

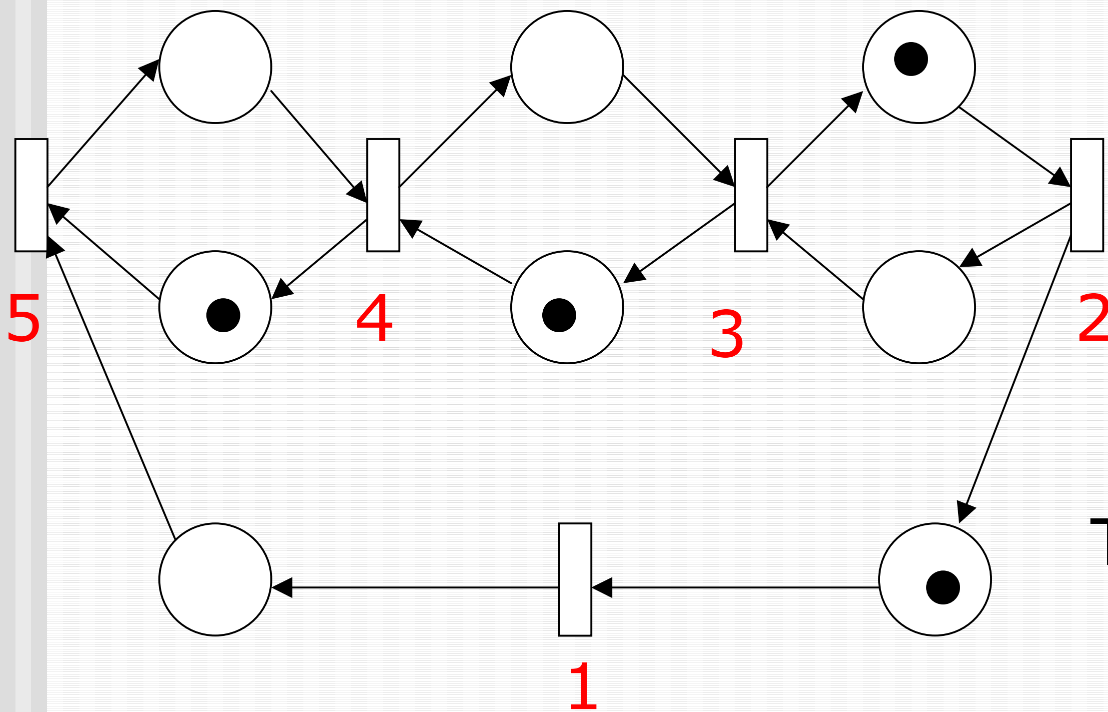
Token Distance

- Token distance matrix T where
  $t_{ij} = \min M_0(P_{ij})$
  or
  ∞ if no directed path P_{ij} exists

  where P_{ij} is path from transition i to transition j

Token Distance (cont)

- Example:



$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Token Distance (cont)

- Useful applications
  - Firability
    - Transition j is firable at a marking M iff all off-diagonal entries of jth column ≥ 0
  - Synchronic distance
    - $d_{ij} = t_{ij} + t_{ji}$
  - Liveness
    - Live iff $t_{ij} + t_{ji} = d_{ij} \neq 0 \ \forall \ i \neq j$

Reference

- Tadao Murata. Petri Nets: Properties, Analysis and Applications. *Proceedings of the IEEE. Vol 77, No 4, April 1989*.

Questions?