Population Dynamics

- Deductive modelling: based on physical laws
- Inductive modelling: based on observation + intuition
- Single species:
  Birth (in migration) Rate, Death (out migration) Rate

\[ \frac{dP}{dt} = BR - DR \]

- Rates proportional to population

\[ BR = k_{BR} \times P; \quad DR = k_{DR} \times P \]

\[ \frac{dP}{dt} = (k_{BR} - k_{DR})P \]
$k_{BR} = 1.4, k_{DR} = 1.2$ : Exponential Growth
\( k_{BR} = 1.4, k_{DR} = 1.2 : \log(\text{Exponential Growth}) \)
\[ k_{BR} = 1.2, \ k_{DR} = 1.4 : \text{Exponential Decay} \]
Logistic Model

• Are \( k_{BR} \) and \( k_{DR} \) really constant?

• Energy consumption in a closed system \( \rightarrow \) limits growth

\[
E_{pc} = \frac{E_{tot}}{P}
\]

\( P \uparrow \rightarrow E_{pc} \downarrow \rightarrow k_{BR} \downarrow \) and \( k_{DR} \uparrow \) until equilibrium

• “crowding” effect:

  ecosystem can support maximum population \( P_{max} \)

\[
\frac{dP}{dt} = k \times \left(1 - \frac{P}{P_{max}}\right) \times P
\]

• crowding is a quadratic effect
\[ k_{BR} = 1.2, \quad k_{DR} = 1.4, \quad crowding = 0.001 \]
Disadvantages

- *NO* physical evidence for model structure!
- But, many phenomena can be well *fitted* by logistic model.
- \( P_{max} \) can only be *estimated* once steady-state has been reached. Not suitable for control, optimisation, . . .
- Many-species system: \( P_{max} \), steady-state?
Multi-species: Predator-Prey

- Individual species behaviour + interactions

- Proportional to species, no interaction when one is extinct: product interaction $P_{\text{pred}} \times P_{\text{prey}}$

$$\frac{dP_{\text{pred}}}{dt} = -a \times P_{\text{pred}} + k \times b \times P_{\text{pred}} \times P_{\text{prey}}$$

$$\frac{dP_{\text{prey}}}{dt} = c \times P_{\text{prey}} - b \times P_{\text{pred}} \times P_{\text{prey}}$$

- Excess death rate $a > 0$, excess birth rate $c > 0$, grazing factor $b > 0$, efficiency factor $0 < k \leq 1$

- Lotka-Volterra equations (1956): periodic steady-state
Predator Prey (population)
Predator Prey (phase)
Competition and Cooperation

• Several species competing for the same food source

\[
\frac{dP_1}{dt} = a \times P_1 - b \times P_1 \times P_2
\]

\[
\frac{dP_2}{dt} = c \times P_2 - d \times P_1 \times P_2
\]

• Cooperation of different species (symbiosis)

\[
\frac{dP_1}{dt} = -a \times P_1 + b \times P_1 \times P_2
\]

\[
\frac{dP_2}{dt} = -c \times P_2 + d \times P_1 \times P_2
\]
Grouping and general \( n \)-species Interaction

- Grouping (opposite of crowding)
  \[
  \frac{dP}{dt} = -a \times P + b \times P^2
  \]

- \( n \)-species interaction
  \[
  \frac{dP_i}{dt} = (a_i + \sum_{j=1}^{n} b_{ij} \times P_j) \times P_i, \forall i \in \{1, \ldots, n\}
  \]

- Only binary interactions, no \( P_1 \times P_2 \times P_3 \) interactions
Forrester System Dynamics

- based on observation + physical insight
- semi-physical, semi-inductive methodology
Methodology

1. levels/stocks and rates/flows

<table>
<thead>
<tr>
<th>Level</th>
<th>Inflow</th>
<th>Outflow</th>
</tr>
</thead>
<tbody>
<tr>
<td>population</td>
<td>birth rate</td>
<td>death rate</td>
</tr>
<tr>
<td>inventory</td>
<td>shipments</td>
<td>sales</td>
</tr>
<tr>
<td>money</td>
<td>income</td>
<td>expenses</td>
</tr>
</tbody>
</table>

2. laundry list: levels, rates, and *causal relationships*

birth rate → birth → population

3. Influence Diagram (+ and -)

4. Structure Diagram (functional relationships)

\[ \frac{dP}{dt} = BR - DR \]
Causal Relationships

- latent variable
- beer consumption
- standard of living (SOL)
- time
- graduates

Hans Vangheluwe  hv@cs.mcgill.ca  Modelling and Simulation: Forrester System Dynamics
Archetypes

- Bellinger http://www.outsights.com/systems/
- influence diagrams
- Common combinations of reinforcing and balancing structures
Archetypes: Reinforcing Loop

state1

state2
Archetypes: Balancing Loop
Forrester System Dynamics

2-species predator–prey system
Inductive Modelling: World Dynamics

• \( BR \): BirthRate
• \( P \): Population
• \( POL \): Pollution
• \( MSL \): Mean Standard of Living
• ...
Inductive Modelling: Structure Characterization

\[ BR = f(P, POL, MSL, \ldots) \]
\[ BR = BRN \times f^{(1)}(P, POL, MSL, \ldots) \]
\[ BR = BRN \times P \times f^{(2)}(POL, MSL, \ldots) \]
\[ BR = BRN \times P \times f^{(3)}(POL) \times f^{(4)}(MSL) \ldots \]

- \( f^{(3)}(POL) \) inversely proportional
- \( f^{(4)}(MSL) \) proportional
- compartmentalize to find correlations
- \ldots Structure Characterization !
Structure Characterisation: LSQ fit

\[ X(t) = -\frac{gt^2}{2} + v_0 t \]

\[ X(t) = A \sin (bt) \]

\[ \text{LSQ (sin)} < \text{LSQ}^2 (t) \]
Feature Extraction

1. Measurement data and model candidates
2. Structure selection and validation
3. Parameter estimation
4. Model use
Feature Rationale

Minimum Sensitivity to Noise
Maximum Discriminating Power
Throwing Stones

Candidate Models

1. \[ x = -\frac{1}{2}gt^2 + v_0 t \]

2. \[ x = Asin(bt) \]
Feature 1 (quadratic model)

\[ g_i = \frac{2x_i}{t_i^2} - \frac{2x_i}{t_i}, i = A, B \]

\[ F1 = g_A / g_B \]
Feature 2 (sin model)

\[ \frac{1}{b} \tan(bt) = \frac{x_i}{\dot{x}_i} \]

solve numerically for \( b \)

\[ F^2 = 200 \frac{|b_A - b_B|}{b_A + b_B} \]
Feature Space Classification

\[ F_1 = \frac{g_A}{g_B} \]
\[ F_2 = \frac{200 |b_A - b_B|}{(b_A + b_B)^{1/b} \tan(b t)} = \frac{x_i}{x_i_{\text{der}}} \]

\[ g_i = \frac{2x_i}{t^2} - \frac{x_{i_{\text{der}}}}{t} \]
Forrester’s World Dynamics model

- “Club of Rome” World Dynamics model
- Few “levels”, note the depletion of natural resources
- implemented in Vensim PLE (www.vensim.com)
World Model Results

6 B Person
10 B Capital units
40 B Pollution units
1e+12 Resource units
2 Satisfaction units

0 Person
0 Capital units
0 Pollution units
0 Resource units
0.4 Satisfaction units

Population : run1 Person
Capital : run1 Capital units
Pollution : run1 Pollution units
Natural Resources : run1 Resource units
quality of life : run1 Satisfaction units
The Process influences Productivity

“Adding manpower to a late software project makes it later”

Fred Brooks. The Mythical Man-Month.

(www.ercb.com/feature/feature.0001.html)

Model in **Forrester System Dynamics**

using Vensim PLE (www.vensim.com)

\[
\text{development rate} = \text{nominal\_productivity} \times (1 - \text{C\_overhead}\times(N\times(N-1)))\times N
\]
Team Size $N = 5$
Team Size $N = 3 \ldots 9$

Optimal Team Size between 7 and 8
The Effect of Adding New Personnel (FSD model)

development rate = nominal_productivity* (1-C_overhead*(N*(N-1)))* (1.2*num_exp_working + 0.8*num_new)
5 New Programmers after 100 days

development rate : n_5
work completed : n_5
work to be completed : n_5
5 New Programmers after 100 days
0 \ldots 6 New Programmers after 100 days

Graph for work to be completed

work to be completed: n_0
work to be completed: n_2
work to be completed: n_5
work to be completed: n_6
Individual-based (discrete-event) model: Productivity.
Individual-based (discrete-event) model: Remaining work.